Stability in Matrix Games

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Main idea

Classical settings. Matrix games and Linear Programming (LP).
Classical question. Stability:

How do our objects of interest change upon perturbations?

Observables. Solutions and value of the problems.

How do solutions and value change upon perturbations?

Matrix Games

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & m_{i,j} & \\ & & \end{pmatrix} \end{array} .$$

$$\text{val} M := \max_{p \in \Delta[m]} \min_{q \in \Delta[n]} p^t M q \,.$$

$$M(\varepsilon) = M_0 + M_1 \varepsilon \,.$$

Derivative of the value function [Mills56]

Define

$$D\mathrm{val}M(0^+) := \lim_{\varepsilon \to 0^+} \frac{\mathrm{val}M(\varepsilon) - \mathrm{val}M(0)}{\varepsilon} .$$

Results.

1. Characterization of $D\mathrm{val}M(0^+)$.
2. (Poly-time) algorithm for computing it.

Theorem ([Mills56])

Given $M(\varepsilon) = M_0 + M_1\varepsilon$,

$$D\mathrm{val}M(0^+) = \mathrm{val}_{P(M_0)\times Q(M_0)} M_1 .$$

Our framework

Polynomial matrix games. Matrix games where payoff entries are given by polynomials.

$$M(\varepsilon) = M_0 + M_1\varepsilon + \ldots + M_K\varepsilon^K .$$

Definition (Value-positivity problem)

$\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0]$ $\quad \mathrm{val} M(\varepsilon) \geq \mathrm{val} M(0)$.

Definition (Uniform value-positivity problem)

$\exists p_0 \in \Delta[m] \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \mathrm{val}(M(\varepsilon); p_0) \geq \mathrm{val} M(0)$.

Definition (Functional form problem)

Return the maps $\mathrm{val} M(\cdot)$ and $p^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Polynomial matrix game

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p^*_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^t \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon} \,.$$

Polynomial matrix game, negative direction

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 2/3$,

$$p_\varepsilon^* = \left(\frac{1-\varepsilon}{2-3\varepsilon}, \frac{1-2\varepsilon}{2-3\varepsilon} \right)^t \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2-3\varepsilon} \,.$$

Types of questions

Question (Complexity)

Determining the complexity of solving these three problems.

Question (Equivalent problem)

Finding the equivalent problems in the setting of Linear Programming.

Algorithms

Lemma (Poly-time algorithms)

There are certifying polynomial-time algorithms for all three problems, for rational data.

Remark

This extends the work of [Mills56].

Equivalence with LPs

Theorem (Adler03)

Matrix games and LPs a poly-time equivalent.

- [Dantzig51] gives an incomplete proof.
- The reduction depends on the computational model: rational, algebraic or real data.

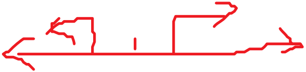
LPs with errors

An error-free LP is the following optimization problem.

$$(P_0)\begin{cases} \min_x & c_0^t x \\ s.t. & A_0 x \leq b_0 \\ & x \geq 0\,, \end{cases}$$

An LP with errors considers polynomials $A(\varepsilon), b(\varepsilon), c(\varepsilon)$.

LP with errors



$$(P_\varepsilon)\begin{cases} \min_x & x \\ s.t. & x \leq -\varepsilon \\ & -x \leq -\varepsilon\,. \end{cases}$$

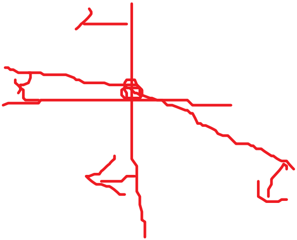
It is not weakly robust, therefore not strongly robust and there is no functional form.

LP with errors

$$
(P_\varepsilon)\begin{cases} \max_{x,y} & x + y \\ s.t. & x \leq 0 \\ & y + \varepsilon x \leq 0\,. \end{cases}
$$

It is weakly robust and strongly robust. Therefore, for $\varepsilon < 1$, the functional form is:

$$
\begin{aligned} \operatorname{val}(P_\varepsilon) &\equiv 0 \\ (x, y)^*(\varepsilon) &\equiv (0, 0)\,. \end{aligned}
$$



Mills in LPs

Theorem ([Mills56])

Under some regularity conditions, the derivative of the value of an LP has a similar characterization as the matrix game.

Asymptotic optimality

Definition (Asymptotically optimal)

A function x^* that is optimal for all $\varepsilon \in (0, \varepsilon_0]$.

Lemma (AFJ13)

There is an EXPTIME algorithm to compute an asymptotically optimal solution of an LP with errors.

Remark

It consists in a variation of the Simplex method.

Reduction

Definition (Weakly robust)

$\exists \varepsilon_0 > 0$ such that, $\forall \varepsilon \in [0, \varepsilon_0]$ (P_ε) is feasible and bounded.

Definition (Strongly robust)

$\exists x^* \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0],$ x^* is also a solution of (P_ε).

Definition (Functional form)

The maps $\mathrm{val}(P_\cdot)$ and $x^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Lemma (Reduction from LP with error to polynomial matrix games)

There is a polynomial-time reduction from robustness problems to the respective value-positivity problem, which preserves the degree of the error perturbation, for algebraic data.

Types of questions (again)

Question (Complexity)

Determining the complexity of solving these three problems.

Question (Equivalent problem)

Finding the equivalent problems in the setting of Linear Programming.

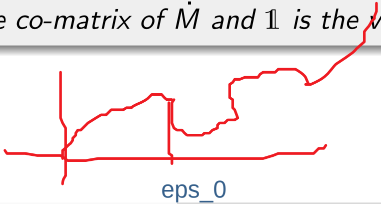
Poly-time for value-positivity

Lemma (Shapley and Snow kernel [SS52])

Let M be a matrix game. There exists a square submatrix $\dot{M}$ such that $\mathbb{1}^t co(\dot{M})\mathbb{1} \neq 0$ and

$$\mathrm{val} M = \frac{\det \dot{M}}{\mathbb{1}^t co(\dot{M})\mathbb{1}},$$

where $co(\dot{M})$ is the co-matrix of $\dot{M}$ and $\mathbb{1}$ is the vector of ones.



LP reduction

Follows from the work of [Adler13]:

- Given an LP with errors, apply the reduction.
- Solving value-positivity problems to the resulting matrix games answers robustness questions.

Comments

Zero-sum discounted stochastic games

Definition (Stochastic game)

A collection of matrix games $M_1, M_2, \ldots, M_K$ together with a transition function q and a discount factor $\lambda \in (0, 1]$. The payoff is given by

$$\lambda \sum_{i \geq 0} (1 - \lambda)^i R_i \,.$$

Towards zero-sum discounted stochastic games

Recall our results about polynomial matrix games.

- [Mills56] presents the idea of iteratively solving a game to find the derivative.
- This is true in more general context than matrix games [MSZ15].
- Perturbation of matrix games has been extended to discounted zero-sum stochastic games [TV80, AFFG01].

Question

Can we extend our analysis to zero-sum discounted stochastic games?

- Parametrized flows
- Derivable matrix games?